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New models of non-stationary processes have been introduced. We showed that the covariance of locally stationary processes can be modeled as pseudodifferential operators. The spectrum of such processes is estimated by approximating the Karhunen-Loeve basis with a local cosine basis, that is optimized with a best basis search algorithm. Applications to geophysics have been studied. Locally dilated processes are a different kind of non-stationary processes that appear in image processing and in physical phenomea that involve Doppler effects. It was shown that the dilation prameters of such processes can be estimated with a wavelet transform, through the solution of a partial differential equation in the scale-space plane. An application concerns the reconstruction of three dimesional surfaces from texture gradient in images. The last part of this project was devoted to the analysis of the distortion-rate function of wavelet image transform codes. By modeling images as element of Besov spaces, we calculated precise analytical formula of distortion-rates, wich are verified by numerical experiments.



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# Estimation with Best Bases

AFOSR grant F49620-96-1-0455
Final Report

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## Abstract

New models of non-stationary processes have been introduced. We showed that the covariance of locally stationary processes can be modeled as pseudo-differential operators. The spectrum of such processes is estimated by approximating the Karhunen-Loeve basis with a local cosine basis, that is optimized with a best basis search algorithm. Applications to geophysics have been studied. Locally dilated processes are a different kind of non-stationary processes that appear in image processing and in physical phenomena that involve Doppler effects. It was shown that the dilation parameters of such processes can be estimated with a wavelet transform, through the solution of a partial differential equation in the scale-space plane. An application concerns the reconstruction of three dimensional surfaces from texture gradient in images. The last part of this project was devoted to the analysis of the distortion-rate function of wavelet image transform codes. By modeling images as element of Besov spaces, we calculated precise analytical formula of distortion-rates, which are verified by numerical experiments.

# 1 Introduction

Most processes encountered in physics or signal processing are non-stationary but are often approximated by stationary processes because of the lack of models and estimation procedures to analyze non-stationarity. In the first part of this project we have introduced a general scheme to estimate the covariance of non-stationary processes by searching for an approximate Karhunen-Loeve basis among a dictionary of bases. This general scheme was applied to locally stationary processes. A model of locally stationary processes based on pseudo-differential operators was introduced. We showed that the spectrum of these locally stationary processes can be efficiently estimated with a dictionary of local cosine bases [1,2]. Applications to geophysics have been studied.

Locally dilated processes is a second important class of non-stationary processes. These processes appear in image textures, in finance and in physical phenomena including Doppler effects. Such processes are created by dilating or compressing a stationary process, with scaling factors that vary with time (or space). The relevant information is often stored in these scaling factors that must be estimated. We proved that the covariance of these dilated processes satisfy a transport differential equation in a scale-space domain [3]. A wavelet based algorithm was derived to estimate the scaling factors.

Wavelet image transform codes are currently the most performant for still image compression, yet there is no theoretical understanding of their distortion rate properties because images are realizations of highly non-stationary processes that are not well understood. We showed that the current distortion-rate theory does not apply to understand the properties of these codes. We introduced a new approach to compute the distortion rate function, by estimating the decay rate of sorted wavelet coefficients. This is equivalent to use models based on Besov spaces.

# 2 Covariance Estimation and Locally Stationary Processes

Second order moments characterize entirely Gaussian processes and are often sufficient to analyze stochastic models, even though the processes may not be Gaussian. When processes are wide-sense stationary, their covariance defines a convolution operator. Many spectral estimation algorithms allow one to estimate the covariance operator from a few realizations, because it is diagonalized with Fourier series or integrals. When processes are not stationary, in the wide-sense, covariance operators may have complicated time varying properties. Their estimation is much more delicate since we do not know a priori how to diagonalize them. The ideas and methods of Calderon and Zygmund in harmonic analysis have shown that although we are not able to find the basis which diagonalizes complicated integral operators in general, it is nevertheless possible to find well structured bases which compress them. This means that the operator is well represented by a sparse matrix with such a basis. This approach allows characterization of large classes of operators by the family of bases which do the compression. We have shown that the ability to represent covariance operators by sparse matrices in a suitable basis leads to its efficient estimation from a few realizations.

In collaboration with Professor Papanicolaou and Dr. Zhang [1] we have concentrated our attention on the class of locally stationary processes, that is, processes whose covariance operators are approximately convolutions. Since cosines and sines diagonalize the covariance of stationary processes, it is natural to expect that *local* cosine functions are "almost" eigenvectors of locally stationary processes. This property is formalized by postulating that the covariance operator is well approximated by a nearly diagonal one in an appropriate local cosine basis. We have shown that if the covariance operator is a pseudo-differential operator of a specified class, then the process is locally stationary.

To estimate the covariance operator of a locally stationary process we search for a local cosine basis which compresses it and estimate its matrix elements. The size of the windows of a suitable local cosine basis must be adapted to the size of the intervals where the process is approximately stationary. Since we do not know in advance the size of approximate stationarity intervals, we have introduced an algorithm that searches within a class of bases for a "best" basis, to compress the covariance operator. This search is done using data provided by a few realizations of the process. It minimizes the Hilbert Schmidt norm of the coefficients along the diagonal of the covariance matrix. One can indeed prove that the Karhunen-Loeve basis is the basis that minimizes this norm. For locally stationary processes, we have derived a fast implementation of the search for a best local cosine basis based on the local cosine trees of Meyer, Coifman and Wickerhauser.

A central issue was to prove that our covariance estimator is consistent when the sample size $N$ increases to infinity. This consistency was obtained with a modified algorithm obtained through a collaboration with Professor Donoho and Dr. von Sachs [2]. The central idea is to regularize the empirical covariance coefficients calculated from the data with a wavelet thresholding algorithm. We proved that for appropriate locally stationary processes, if the size of the maximum eigenvalue is bounded then the best basis covariance estimator converges to the true covariance.

# 3 Locally Dilated Processes

Locally dilated processes are obtained with a smooth change of variable $\gamma(t)$ of a stationary process, with possibly an amplitude modulation $a(t)$

$$Y(t) = \alpha(t)\, X(\gamma(t)).$$

The derivative $\gamma'(t)$ can be interpreted as a time varying scaling factor. Such processes appear in image processing and number of physical problems that incorporated Doppler effects. In image processing, a homogeneous texture that can be modeled with a stationary process $X$ will appear as locally dilated in the image because of the perspectivity effect. This is called a "texture gradient". Recovering this local dilation allows one to compute the three dimensional coordinates of the surfaces visualized in the scene, up to a multiplication factor. In collaboration with Maureen Clerc [3], we studied the estimation of $\alpha(t)$ and $\gamma(t)$ from a single realization of $Y(t)$.

We suppose that the variations of $\alpha(t)$ and $\gamma(t)$ happens on a larger scale than the fine scale variations of $X(t)$. We define the covariance

$$C(u,v) = E\{Y(u+\frac{v}{2})\, Y(u-\frac{v}{2})\}.$$

The scale separation hypothesis allows one to prove that $C(u,v)$ nearly satisfies a differential equation

$$\frac{\partial \log C(u,v)}{\partial u} = \frac{d\log\alpha(u)}{du} + \frac{d\log\gamma'(u)}{du}\,\frac{\partial \log C(u,v)}{\partial \log v}. \tag{1}$$

This equation can be interpreted as a transport equation in scales where $\Big(\log\gamma'(u)\Big)'$ is a velocity along scales. We show that we get nearly optimal statistical estimators of $\frac{\partial \log C(u,v)}{\partial u}$ and $\frac{\log \partial C(u,v)}{\partial \log v}$ from a single realization of $Y(t)$ by computing the covariance in a wavelet basis at fine scales [3]. The wavelet transform of a signal $f$ is defined by

$$Wf(b,a) = \int_{-\infty}^{+\infty} f(t)\frac{1}{\sqrt{a}}\psi(\frac{t-b}{a})dt.$$

The variance of the wavelet transform of $X(t)$ is $R(b,a) = E\{WY(b,a)\}$. At fine scale, we proved that the wavelet transform satisfies the a transport equation similar to (1)

$$\frac{\partial \log R(u,a)}{\partial a} = \frac{d \log \alpha(u)}{du} + \frac{d \log \gamma'(u)}{du} \frac{\partial \log R(u,a)}{\partial \log a}. \tag{2}$$

An iterative algorithm has been designed to estimate $\gamma(t)$ by solving (2) from the empirical estimates of the wavelet transform covariance $R(u,a)$ [3]. This algorithm has been implemented and tested numerically on one-dimensional signals.

## 4 Understanding Image Transform Codes

Research on image compression led to a culture shock between several mathematicians who got interested in this topic through the discovery of wavelet orthonormal bases, and signal processing engineers that had developed efficient algorithms for many years. If one can approximate an image with few wavelet image coefficients, many mathematicians thought that these bases should be remarkably well adapted to perform bit compression. Most signal processors have replied that the problem is much more difficult since it also involves a careful analysis of the performance of scalar quantizations and bit allocations. If the signals to be encoded are realizations of a Gaussian process, under the high resolution quantization hypothesis, we know nearly everything on the performance of a transform code. For an average of $\bar{R}$ bits per pixel, the mean-square error $D(\bar{R})$ decays proportionally to $2^{-2\bar{R}}$ with a constant that depends upon the bit allocation and the basis. The optimal basis is then the Karhunen-Loeve basis, and wavelet bases are a priori not good approximations of Karhunen-Loeve bases for images. It is known that images are not realizations of Gaussian processes, but it is often believed that the use of Gaussian and high resolution approximations give an appropriate estimate of the distortion-rate curve $D(\bar{R})$. Hence the widespread use of the coding "gain" formula which is only justified in this framework.

Current image transform codes operate below 1 bit per pixel. For such high compression rates, we have shown that the classical transform code theory yields a wrong estimate of the distortion-rate $D(\bar{R})$ [4,5]. If one optimizes the scalar quantization and bit allocation then $D(\bar{R})$ depends mostly on the precision of signal approximations with few non-zero decomposition coefficients in the orthogonal basis. Yet, as emphasized by signal processors, there are important constants involved, that depend upon the encoding efficiency of the positions of zero coefficients. At low bit rates, for a number of bit per pixel $\bar{R}$ that is below 1, we proved [4,5] that

$$D(\bar{R}) \approx (1+K) D_0(\frac{\bar{R}}{r}) \sim \bar{R}^{1-2\alpha}, \tag{3}$$

where $K$ and $r$ are two constants that are specified, and $D_0(z)$ is the error when approximating a signal with a proportion of $z$ non-zero basis coefficients. For natural images decomposed in wavelet or cosine bases, the parameter $\alpha$ is typically of the order of 1. It is related to the exponent of the smallest Besov space that includes the signal.

Transform code algorithms can be improved by an embedding strategy which sends first the larger amplitude coefficients and progressively refine their quantization. The improvement provided by the partial ordering of embedded codes has been analyzed mathematically and evaluated numerically for wavelet zero-trees [4,5].

# 5 Technical Publications

1. S. Mallat, G. Papanicolaou, Z. Zhang, "Adaptive Covariance Estimation of Locally Stationary Processes", to appear in the Annals of Statistics.

2. D. Donoho, S. Mallat, R. Von Sachs, "Estimating Covariances of Locally Stationary Processes: Consistency of Best Basis Methods", submitted to the Annals of Statistics.

3. M. Clerc, S. Mallat, "Estimation of locally dilated processes", *IMS International conference*, South Lake City, July 1997.

4. S. Mallat, F. Falzon, "Understanding wavelet image transform codes", *SPIE conference on Aerospace,* Orlando, April 1997.

5. S. Mallat and F. Falzon, "Understanding image transform codes", to appear in *IEEE Transactions on Signal Processing,* January 1998.

# 6 Awards

Stéphane Mallat:

- *Outstanding Achievement Award*, awarded by the SPIE Optical Engineering Society, 1997.

- *Blaise Pascal Prize for Applied Mathematics*, awarded by the French Academy of Sciences, 1997.